Limited Use Copyright

1. J. Weston, Walch, Publisher, grants to Srikant Nekkanty a limited, one-time right to scan pages 75-77 of "Amusement Park Physics" onto a secure server for student instruction.

2. No copies of "Amusement Park Physics" may be printed from the secure server. If and when needed, Srikant Nekkanty will purchase additional copies of "Amusement Park Physics" from Walch Education.

3. Srikant Nekkanty will scan this limited use language and place it at the beginning part of the scanned material, to ensure that teachers and students understand the limited right of use.

4. If Walch Education determines for any reason to suspend this right, Srikant Nekkanty will promptly remove the scanned material from the secure server.
CLOTHOID LOOPS

When most people first view a looping roller coaster, they think that the loop is a circle. This is a common misconception. These vertical loops follow a special shape called a clothoid (also spelled klothoid). The clothoid loop is actually a section of a cornu spiral. Cornu spirals have applications in Fresnel diffraction problems and in the design of highway exit ramps. Figure 37 shows the clothoid loop, which is a cornu spiral section with its reflection, and resembles an upside-down teardrop.

The design most commonly used in amusement parks involves a typical linear approach to the bottom of the loop, followed by a curve with a regularly decreasing radius. With respect to a vertical line bisecting the loop, the top +65° approximates the arc of a circle. Specifically, the curvature of the clothoid is proportional to the arc length at any point on the curve. That is to say,

\[ \frac{1}{r} = \frac{s}{a^2} \]

where:

- \( a \) is some constant that determines the tightness of the curve,
- \( r \) is the radius at the point in question, and
- \( s \) is the arc length (see Figure 38).

Safety and passenger comfort are the primary reasons for use of the clothoid loop. Normally in circular motion, if all other things are kept constant, as the radius reduces, the tangential velocity increases. We assume there is negligible friction here. A classic example of this is ice skaters or ballet dancers going into a tight pirouette, spinning faster as their arms and legs are brought in and more slowly as they extend their extremities. But the situation with the roller coaster is more complicated than that. As the roller coaster climbs, its energy flows from the kinetic energy of the moving train to the gravitational potential energy stored in the gravitational field, and therefore reduces its velocity. If the loop were a circle, there would be a diminishing centripetal acceleration as the coaster reduces its speed due to its change in height. If a clothoid track is chosen, the centripetal acceleration increases as you go up into the loop. So two things are at play here on the way up to the top of the loop: the increase of speed due to the decreasing radius of the loop, and the decrease of tangential velocity due to the decrease of kinetic energy.

\[ AB \] is approximately linear
\[ BC \] is a clothoid
\[ CD \] is circular
\[ OA \] height = 16.3 m
\[ \theta = 15^\circ \]

Figure 38
Curvature of Clothoid
With this background, we only begin to understand the problem. We still have not answered the question, "Why not just use a circular track?" Most people begin to feel uncomfortable at sustained accelerations in excess of 3.5 g's. Beyond 5 g's, many people begin to gray out and lose consciousness. From the amusement park's point of view (and that of many passengers), this is an undesired side-effect! The derivation that follows shows that if a circular loop were used, the minimum acceleration at the bottom of a circular loop would be in the range of 6 g's—not good.

Fortunately, physics comes to the rescue. If we start with a curve that is greater than what is needed, our change in velocity will be smaller. This leads to two problems—enormous circles (not enough room and too expensive to build) and running out of kinetic energy before reaching the top of the loop (thus, not going upside down). As the tangential velocity falls due to climbing, we can counteract that by tightening the curvature of the circle. Thus, decreasing tangential velocity due to the loss of kinetic energy (the climb) is countered with the increase in tangential velocity due to the reduced radius. The clothoid shape satisfies these conditions so that there is enough kinetic energy to go over the top (upside down) with accelerations at the bottom that rarely exceed 3.7 g's.

The descending trip is just the reverse of the ascending trip. As we increase our tangential velocity due to falling, the curvature of the loop is increased so that the resulting acceleration when hitting the bottom of the loop is within acceptable limits. In short, the change in tangential velocity due to the change in radius of the curve is offset by the change in tangential velocity due to height.

Investigating a frictionless circular loop of radius R with the condition that the train just barely remain on the track at the top of the loop, we find that the gravitational force must be equal to the centripetal force due to the circular motion. See Figure 39.

![Figure 39](image)

**Force Diagram at Top of Circular Loop**

Solving, using Newton's second law, we have:

$$\sum F = F_{net} = ma = \frac{mv_{top}^2}{r} = F_s - mg.$$

Since the force of support from the track is zero, in this case, we have:

$$\frac{mv_{top}^2}{r} = mg.$$

Solving for the tangential velocity at the top, we have:

$$v_{top}^2 = rg.$$

The gravitational potential energy at the top of the circular loop is:

$$GPE = mg2r,$$

since the diameter of the circle, or 2r, is the height climbed. The total energy at the top is the gravitational potential energy due to the change in height and the kinetic energy just moving the train over the top. Hence:

$$E_{total} = GPE + KE$$

$$= mgAy + \frac{1}{2}mv^2.$$

Substituting 2r for the height and rg for $v^2$

$$= mg2r + \frac{1}{2}m(rg)$$

$$= 2.5mgr.$$
In a friction-free system, this must be equal to the gravitational potential energy of the train on the lift hill at elevation \(Ay\), before entering the circular loop; hence:

\[ mgAy = 2.5mgR. \]

Therefore, the minimum height of the starting hill that drops the roller coaster into the circular loop must be 2.5\(R\) of the circular loop.

Again, assuming a friction-free system, by conservation of energy, the gravitational potential energy at the top of the lift hill must be equal to the kinetic energy at the bottom of the loop:

\[
mgAy = 2.5mgR = \frac{1}{2}mv^2
\]

(Solving for \(v^2\), we have:

\[ v_{bottom} = 5g. \]

At the bottom, the track must not only support the train, but also keep the train moving with circular motion (\(mv^2/r\text{bottom}\)). The forces at the bottom must be:

\[
\sum F = F_{net} = m\alpha = F_{support} = mg
\]

\[
\frac{mv^2}{r} = F_{support} - mg
\]

\[
F_{support} = \frac{mv^2}{r} + mg
\]

Substituting for \(v^2\),

\[
F_{support} = \frac{5mgR}{r} + mg + mg + mg = 6mg
\]

Simplifying, \(a = 6g\) at the bottom.

In reality, this situation does not exist, since we have ignored frictional effects. Clearly, an even greater velocity, and thus greater acceleration, would be necessary; this could be obtained from a taller lift hill or catapulting the train. This would increase the acceleration at the bottom, entering the loop at an acceleration of over 6\(g\)'s. This large acceleration is unacceptable.

Thus, the elegance of the clothoid loop. Due to its changing radius of curvature, smaller accelerations are experienced at the bottom than if a circular track were used.